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# Research in Heterogeneous and Nonlinear Media

*Final technical report on AFOSR AASERT grant no. F49620-92-J-0272*

*Robert V. Kohn, Principal Investigator*

*June 1, 1992 - August 31, 1995*

## SUMMARY

This AASERT grant was awarded in affiliation with AFOSR grant 90-0090. During 1992-3 and 1993-4 the student supported by this grant was Ken McLaughlin. During 1994-5 the student supported by this grant was Ivan Blank. Both McLaughlin and Blank are U.S. citizens. McLaughlin is now an Assistant Professor at Ohio State University. Blank is continuing his graduate study.

### 1. Research of Ken McLaughlin.

The first two years of this AASERT grant supported the research of Ken McLaughlin concerning the continuum limit of the Toda lattice. Ken earned his a Ph.D. in August, 1994 [3], and assumed an Assistant Professorship at Ohio State University in September 1994. His thesis work was recently submitted for publication [1].

The Toda lattice is this system of  $2n - 1$  differential equations:

$$\frac{d}{dt}a_k = 2(b_k^2 - b_{k-1}^2) \quad k = 1, \dots, n \quad \frac{d}{dt}b_k = b_k(a_{k+1} - a_k) \quad k = 1, \dots, n-1. \quad (1)$$

It is a completely integrable Hamiltonian system, and can be solved by the methods of inverse spectral theory. If we formally set

$$a\left(\frac{k}{n+1}, t\right) = a_k(t') \quad k = 1, \dots, n, \quad b\left(\frac{k}{n+1}, t\right) = b_k(t') \quad k = 1, \dots, n-1$$

with  $t' = (n+1)t$ , then in the continuum limit ( $n \rightarrow \infty$ ) one gets a system of nonlinear hyperbolic equations for the functions  $a(x, t)$  and  $b(x, t)$

$$\frac{\partial a}{\partial t} = 2 \frac{\partial (b^2)}{\partial x}, \quad \frac{\partial b}{\partial t} = b \frac{\partial a}{\partial x}. \quad (2)$$

McLaughlin's thesis made this correspondence rigorous up to the time when the solution of (2) develops a shock, and explored the behavior of the continuum limit after shock formation. He studied the behavior near the time of shock formation in detail. He also studied the long-time asymptotics, with and without shocks.

The importance of this work lies in its implications for the convergence (and non-convergence) of numerical approximation schemes. The discrete system (1) is essentially a dispersive approximation of the hyperbolic system (2). When the PDE develops a shock, this type of numerical scheme is expected to develop oscillations [4]. One important conclusion of

McLaughlin's work is the insight that *this is not the only possibility*: in some cases the discrete system may converge strongly even after shock formation. When oscillations do develop, one can describe them in detail by taking advantage of the completely integrable character of (1). The point, of course, is that the general features of the solution may be typical even of non-integrable approximation schemes.

Computer simulation played a significant role in this project. It was used to identify classes of initial data with specific continuum limits. Then analytical methods were applied to give rigorous proofs of the expected limiting behavior.

We now give a little more detail concerning the nature of McLaughlin's results. The first task was to derive detailed information about the solution of (1) in terms of its initial data. This was accomplished using methods from inverse spectral theory.

The second task was to prove convergence prior to shock formation. This was accomplished using methods similar to those developed by Lax and Levermore for their work on the small-dispersion limit of the Korteweg-deVries equation [2]. A class of data was produced for which McLaughlin proved global existence, i.e. that singularities never form in the PDE. For this data it makes sense to consider the long-time asymptotics of the discrete system. McLaughlin obtained rigorous results of this type, by adapting the methods of [2].

The third and most interesting task was to study the continuum limit after shock formation. McLaughlin has done this for selected classes of initial data, leading to increasingly complex behavior after the PDE develops a shock. He identified one class of data for which the solutions of the ODE system (1) converge strongly even *after* the shock time. This demonstrates that dispersive numerical approximation schemes with boundary conditions may sometimes approximate shock-formation without introducing oscillations.

McLaughlin also studied a second class of data for which the PDE develops a shock and the ODE develops sustained oscillations. Near the shock time these oscillations are restricted to a spatial interval emanating from the shock. On the complement of this interval, the ODE solutions converge smoothly to those of the PDE. The endpoints of this interval can be determined from the solution of the PDE; they may be viewed as marking a "phase transition" whereby space-time is separated into two regions, one characterized by oscillations and the other by strong convergence. In the oscillatory region, the continuum limit is no longer described by the PDE system (2). Near the singular time it is described by a different hyperbolic system in 4 unknowns. In general there can be regions of "multiphase oscillations," whose continuum limit is described by larger hyperbolic systems. But McLaughlin's thesis identifies a class of initial data for which multiphase oscillations do not occur. Similar results for KdV were obtained by Tian [5].

The modifications to the method of Lax and Levermore presented by McLaughlin permit rigorous long-time asymptotics of the continuum limit of the ODE system in the presence of oscillations, independent of the nature of the oscillations, whether they be single or multi phase.

Looking toward the future, this work highlights an important direction for research on nonintegrable dispersive approximation schemes. Formal asymptotics exist to describe the oscillations arising from such approximations [4], but the locus of the "phase transition" separating weak and strong convergence has not yet been studied.

## 2. Research of Ivan Blank

The final year of this AASERT grant supported the study of Ivan Blank during the academic year 1994-5. Blank was a second-year student when he received this support. He therefore took a full load of graduate-level courses, and spent much time preparing for his Oral Preliminary Examinations. His special topic for the Oral Exam was Partial Differential Equations. Blank passed this exam in October, 1995, and he is now about to start a focussed research project.

## References

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